

Gibbs Measures on Marked Configuration Spaces: Existence and Uniqueness

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1 Introduction

The aim of this paper is to study the equilibrium states of following infinite particle system in continuum. We consider a countable collection γ of identical point particles chaotically distributed over a Euclidean space X ($= \mathbb{R}^d$). Additionally, we assume that each particle $x \in \gamma$ possesses internal structure described by a mark (spin) σ_x taking values in a single-spin space S ($= \mathbb{R}^m$) and characterized by a single-spin measure g on S . Each two

particles $x, y \in \gamma$ interact via a pair potential Ψ given by the sum of two components:

(i) a purely positional (e.g., distance dependent, possibly singular or hard-core) background potential

$$\Phi : X \times X \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \Phi(x, y) = \Phi(y, x), \quad x, y \in X; \quad (1)$$

(ii) a (position dependent) spin-spin interaction of the form $J_{xy}W(\sigma_x, \sigma_y)$, where

$$J : X \times X \rightarrow \mathbb{R} \text{ and } W : S \times S \rightarrow \mathbb{R} \quad (2)$$

are symmetric functions.

For technical reasons we suppose that the interaction has finite range, i.e., there exists $R > 0$ such that $\Phi(x, y) = 0$ and $J_{xy} = 0$ if $|x - y| > R$.

The whole system is then governed by the heuristic Hamiltonian

$$\hat{H}(\hat{\gamma}) := \sum_{\{x, y\} \subset \gamma} \Phi(x, y) + \sum_{\{x, y\} \subset \gamma} J_{xy}W(\sigma_x, \sigma_y)$$

on the phase space $\Gamma(X, S)$ consisting of marked configurations $\hat{\gamma} = \{(x, \sigma_x)\}$, where the corresponding position configuration $\{x\}$ belongs to the space

$$\Gamma(X) := \{\gamma \subset X : N(\gamma_\Lambda) < \infty \text{ for any } \Lambda \in \mathcal{B}_c(X)\}. \quad (3)$$

Here $\mathcal{B}_c(X)$ is the collection of all compact subsets of X and $N(\gamma_\Lambda)$ denotes the number of elements of $\gamma_\Lambda := \gamma \cap \Lambda$. In what follows, we will use the notation $\hat{\gamma}_\Lambda := \{(x, \sigma_x), x \in \gamma_\Lambda\}$.

The equilibrium states of the system are described by certain probability measures $\Gamma(X, S)$. In absence of the interaction, the equilibrium state is unique and given by the marked Poisson measure

$$\hat{\pi}(d\hat{\gamma}) = \bigotimes_{x \in \gamma} g(d\sigma_x) \pi(d\gamma),$$

where π is the Poisson measure on $\Gamma(X)$ with intensity (particle density) $z > 0$, see [10]. If the interaction is present, the equilibrium states are given by marked Gibbs measures μ on $\Gamma(X, S)$ that are constructed as perturbations of $\hat{\pi}$ by the (heuristic) density $\exp\left(-\beta \hat{H}(\hat{\gamma})\right)$, where $\beta > 0$ is the inverse temperature of the system. Rigorously, any such μ is a probability measure on $\Gamma(X, S)$ with prescribed conditional distributions $\mu(d\hat{\gamma} | \hat{\gamma}_{X \setminus \Lambda} = \hat{\eta})$, $\hat{\eta} \in \Gamma(X \setminus \Lambda, S)$, for a system of sets $\Lambda \in \mathcal{B}_c(X)$ exhausting X , which are given by the Gibbs specification kernels $\Pi_\Lambda(d\hat{\gamma} | \hat{\eta})$ of our model (see formulae (14),

(15)). This constitutes the standard Dobrushin-Lanford-Ruelle formalism described in details in Section 2.2.

We denote by \mathcal{G} the set of all such measures (for fixed Ψ and g). The study of the structure of the set \mathcal{G} is of a great importance. In particular, there are three fundamental questions arising here:

- (E) *Existence*: is \mathcal{G} not empty?
- (U) *Uniqueness*: is \mathcal{G} a singleton?
- (M) *Multiplicity*: does \mathcal{G} contain at least two elements?

In this paper, we derive sufficient conditions for (E) and (U). We introduce the set $\mathcal{G}^t \subset \mathcal{G}$ tempered Gibbs measures, which are concentrated on the space $\Gamma^t(X, S)$ of configurations with certain bounds on their density, see (25), (26). Under reasonable stability assumptions on the interaction potentials Φ and W , we will prove that the set \mathcal{G}^t is not empty (Theorem 2) and, moreover, that \mathcal{G}^t is a singleton provided the couplings J_{xy} and the particle density z are small enough (Theorem 3). To prove the existence, we use the extension of the method developed in [21] for the case of interacting particles without spins. The crucial technical step here is to prove uniform bound of certain exponential moments of the corresponding specification kernels $\Pi_\Lambda(d\hat{\gamma}|\hat{\eta})$ for any tempered boundary condition $\hat{\eta}$. This in turn allows to show the compactness (in the topology of local set convergence on $\Gamma(X, S)$) of the family $\{\Pi_\Lambda(d\hat{\gamma}|\hat{\eta}), \Lambda \in \mathcal{B}_c(X)\}$ and thus the existence of the limiting points, which can be identified with elements of \mathcal{G}^t .

In order to study the uniqueness, we represent the configuration space $\Gamma(X, S)$ in the form $\Gamma(Q, S)^{\mathbb{Z}^d}$, where Q is a cube in X , via the natural embedding $\mathbb{Z}^d \subset X$, and use the result of [5, Theorem 2.6], which develops the Dobrushin-Pechersky approach to the uniqueness problem for lattice-type systems [12], see also [33, Theorem 4] and [2, Theorem 3], where this method is applied to continuum systems (without spins). The uniform exponential moment bounds allow us to control the interaction growth and to check the conditions of the Dobrushin-Pechersky theorem. As a by-product of our method we also prove a decay of correlations for the (unique) Gibbs measure (Corollary 4), which seems to be entirely new for such systems.

Let us note that the general theory of Gibbs measures with the Ruelle-type (super-) stable interactions on marked configuration spaces can be found e.g. in [23], [1], [19] and [29]. However, it is essentially restricted to bounded spins and hence does not apply to our model (see Remark 6). The case of unbounded spins and non-attractive interactions has not been treated so far in the literature.

We note that Question (M) has been discussed in the complementary paper [7].

The structure of the paper is as follows. In Section 2 we give the rigorous description of our model (Subsections 2.1, 2.2, 2.3) and formulate the main results (subsection 2.4). Section 3 is devoted to the derivation of the moment bounds. In Section 4, we prove our main existence result - Theorem 2. Section 5 deals with the uniqueness problem. We start with the lattice representation of our model (Subsection 5.1) and prove Theorem 3 in Subsection 5.2. In Appendix, we present proofs of several technical lemmas.

2 The model and main results

2.1 Marked configuration spaces

Consider the configuration space $\Gamma(X)$ defined by formula (3). It can be endowed with the vague topology, which is the weakest topology that makes continuous all mappings

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_0(X).$$

It is known that this topology is completely metrizable, which makes $\Gamma(X)$ a Polish space (see, e.g., [14, Section 15.7.7] or [35, Proposition 3.17]). An explicit construction of the appropriate metric can be found in [20]. By $\mathcal{P}(\Gamma(X))$ we denote the space of all probability measures on the Borel σ -algebra $\mathcal{B}(\Gamma(X))$ of subsets of $\Gamma(X)$.

Let us now consider the product space $X \times S$, where $S = \mathbb{R}^m$ is another Euclidean space. The canonical projection $p_X : X \times S \rightarrow X$ can be naturally extended to the configuration space $\Gamma(X \times S)$. Observe that for a configuration $\hat{\gamma} \in \Gamma(X \times S)$ its image $p_X(\hat{\gamma})$ is a subset of X that in general admits accumulation and multiple points. The marked configuration space $\Gamma(X, S)$ is defined in the following way:

$$\Gamma(X, S) := \{\hat{\gamma} \in \Gamma(X \times S) : p_X(\hat{\gamma}) \in \Gamma(X)\}. \quad (4)$$

The space $\Gamma(X, S)$ will be endowed with a (completely metrizable) topology defined as the weakest topology that makes the maps

$$\Gamma(X, S) \ni \hat{\gamma} \mapsto \langle f, \hat{\gamma} \rangle \quad (5)$$

continuous for all bounded continuous functions $f \in X \times S \rightarrow \mathbb{R}$ such that $\text{supp} f \subset \Lambda \times S$, for some $\Lambda \in \mathcal{B}_c(X)$, i.e. with spatially compact support.

This topology has been used in e.g. [1], [6] and [23]. In what follows, we will call it the τ -topology. Notice that $(\Gamma(X, S), \tau)$ is a Polish space, cf. Section 2 in [6], where a concrete metric that generates the topology τ is given. We equip $\Gamma(X, S)$ with the corresponding Borel σ -algebra $\mathcal{B}(\Gamma, S)$.

Along with $\Gamma(X, S)$ we will also use the spaces $\Gamma(\Lambda, S)$, $\Lambda \in \mathcal{B}_0(X)$, and the space $\Gamma_0(X, S) := \bigcup_{\Lambda \in \mathcal{B}_0(X)} \Gamma(\Lambda, S)$ of finite marked configurations, endowed with the Borel σ -algebras $\mathcal{B}(\Gamma(\Lambda, S))$ and $\mathcal{B}(\Gamma_0(X, S))$ respectively, which are induced by the Euclidean structure of X . It is known that $\mathcal{B}(\Gamma_0(X, S)) = \{A \cap \Gamma_0(X, S) : A \in \mathcal{B}(\Gamma(X, S))\}$.

The spaces $\Gamma(\Lambda, S)$ and $\Gamma_0(X, S)$ can be identified with the corresponding subspaces of $\Gamma(X, S)$ via the natural embedding. Clearly, these subspaces belong to $\mathcal{B}(\Gamma(X, S))$ and σ -algebras $\mathcal{B}(\Gamma(\Lambda, S))$ and $\mathcal{B}(\Gamma_0(X, S))$ can be considered as sub-algebras of $\mathcal{B}(\Gamma(X, S))$.

On the other hand, we can introduce the algebras $\mathcal{B}_\Lambda(\Gamma(X, S))$ of sets $C_B := \{\gamma \in \Gamma(X) : \gamma_\Lambda \in B\}$, $B \in \mathcal{B}(\Gamma(\Lambda, S))$ and define the algebra of local (cylinder) sets

$$\mathcal{B}_{\text{loc}}(\Gamma(X, S)) := \bigcup_{\Lambda \in \mathcal{B}_0(X)} \mathcal{B}_\Lambda(\Gamma(X, S)). \quad (6)$$

In a similar way, one introduces the spaces $\Gamma(\Lambda)$, $\Gamma_0(X)$ and the corresponding algebras $\mathcal{B}(\Gamma(\Lambda))$, $\mathcal{B}(\Gamma_0(X))$ and $\mathcal{B}_{\text{loc}}(\Gamma(X))$.

Observe that the space $(\Gamma(X, S), \mathcal{B}(\Gamma(X, S)))$ can be obtained as a projective limit of spaces $(\Gamma(\Lambda, S), \mathcal{B}(\Gamma(\Lambda, S)))$, $\Lambda \in \mathcal{B}_c(X)$, with respect to projection maps

$$p_{\Lambda_2, \Lambda_1} : \Gamma(\Lambda_2, S) \ni \hat{\gamma} \mapsto \hat{\gamma}_{\Lambda_1} := (\gamma_{\Lambda_1}, \sigma_{\gamma_{\Lambda_1}}) \in \Gamma(\Lambda_1, S). \quad (7)$$

The space $\Gamma(X, S)$ has the structure of a fibre bundle over $\Gamma(X)$ with fibres $p_X^{-1}(\gamma)$, which can be identified with the product spaces

$$S^\gamma = \prod_{x \in \gamma} S_x, \quad S_x = S.$$

Thus each $\hat{\gamma} \in \Gamma(X, S)$ can be represented by the pair

$$\hat{\gamma} = (\gamma, \sigma_\gamma), \quad \text{where } \gamma = p_X(\hat{\gamma}) \in \Gamma(X), \quad \sigma_\gamma = (\sigma_x)_{x \in \gamma} \in S^\gamma.$$

It follows directly from the definition of the corresponding topologies that the map $p_X : \Gamma(X, S) \rightarrow \Gamma(X)$ is continuous. Thus for any configuration γ the space S^γ is a Borel subset of $\Gamma(X, S)$.

Given a set $A \subset X$ we will systematically use the notation $\hat{A} := A \times S$.

2.2 The model

In this section, we will give the rigorous definition of Gibbs measures associated with the interaction potentials (1), (2) and the single-spin measure g , and introduce the conditions that guarantee $\mathcal{G} \neq \emptyset$.

We define the energy function $\hat{H} : \Gamma_0(X, S) \rightarrow \mathbb{R}$ by the formula

$$\hat{H}(\hat{\gamma}) := H(\gamma) + E_\gamma(\sigma), \quad \hat{\gamma} \in \Gamma_0(X, S), \quad (8)$$

with

$$H(\gamma) = \sum_{\{x,y\} \subset \gamma} \Phi(x, y) \text{ and } E_\gamma(\sigma) = \sum_{\{x,y\} \subset \gamma} J_{xy} W(\sigma_x, \sigma_y). \quad (9)$$

For any $\Delta \in \mathcal{B}_c(X)$ define the relative local interaction energy

$$\hat{H}_\Delta(\hat{\gamma}_\Delta | \hat{\eta}) := H_\Delta(\gamma_\Delta | \eta) + E_{\gamma_\Delta, \eta}(\sigma | \xi), \quad \hat{\eta} = (\eta, \xi) \in \Gamma(X, S), \quad (10)$$

where

$$H_\Delta(\gamma_\Delta | \eta) = H(\gamma_\Delta) + \sum_{x \in \gamma_\Delta} \sum_{y \in \eta_{\Delta^c}} \Phi(x, y)$$

and

$$E_{\gamma_\Delta, \eta}(\sigma_{\gamma_\Delta} | \xi) = E_{\gamma_\Delta}(\sigma_{\gamma_\Delta}) + \sum_{x \in \gamma_\Delta} \sum_{y \in \eta_{\Delta^c}} J_{xy} W(\sigma_x, \xi_y). \quad (11)$$

We assume that the single-spin measure g on S has the form $g(ds) := e^{-V(s)} ds$, where $V : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable (bounded from below) function.

For $\Delta \in \mathcal{B}_c(X)$ we introduce the local Gibbs state at inverse temperature $\beta > 0$ by the formula

$$\mu_\Delta^{\hat{\eta}}(d\hat{\gamma}_\Delta) := Z_\Delta(\hat{\eta})^{-1} \exp \left(-\beta \hat{H}_\Delta(\hat{\gamma}_\Delta | \hat{\eta}) \right) \hat{\lambda}(d\hat{\gamma}_\Delta) \in \mathcal{P}(\Gamma(\Delta, S)), \quad (12)$$

where $\hat{\lambda}(d\hat{\gamma}_\Delta) := \bigotimes_{x \in \gamma_\Delta} g(d\sigma_x) \lambda(d\gamma_\Delta)$ and λ is the Lebesgue-Poisson measure (with intensity dx and activity parameter z) on $\mathcal{B}_{loc}(\Gamma(X))$ (see e.g. [21], [23]). Here

$$Z_\Delta(\hat{\eta}) := \int \exp \left(-\beta \hat{H}_\Delta(\hat{\gamma}_\Delta | \hat{\eta}) \right) \hat{\lambda}(d\hat{\gamma}_\Delta) \quad (13)$$

is the normalizing factor (called the partition function) making $\mu_\Delta^{\hat{\eta}}$ a probability measure on $\Gamma(\Delta, S)$ (provided $Z_\Delta(\hat{\eta}) \neq 0$, which will be the case under certain conditions on the interaction potentials, cf. Proposition 12). Next, we can define a Gibbsian specification kernel $\Pi_\Delta(\cdot | \hat{\eta}) \in \mathcal{P}(\Gamma(X, S))$ by the formula

$$\Pi_\Delta(B | \hat{\eta}) := \mu_\Delta^{\hat{\eta}}(B_{\Delta, \hat{\eta}}), \quad (14)$$

where $B_{\Delta, \hat{\eta}} := \{\hat{\gamma} \in \Gamma(X, S) : \hat{\gamma}_{\Delta} \cup \hat{\gamma}_{\Delta^c} \in B\} \in \mathcal{B}(\Gamma(\Delta, S))$, $B \in \mathcal{B}(\Gamma(X, S))$, so that the following integral relation holds:

$$\begin{aligned} \int_{\Gamma(X, S)} F(\hat{\gamma}) \Pi_{\Delta}(d\hat{\gamma} | \hat{\eta}) &= Z_{\Delta}(\hat{\eta})^{-1} \int_{\Gamma(X, S)} F(\hat{\gamma}_{\Delta} \cup \hat{\gamma}_{\Delta^c}) \\ &\times \exp(-\beta H_{\Delta}(\gamma_{\Delta} | \eta) - \beta E_{\gamma_{\Delta} \cup \eta_{\Delta^c}}(\sigma_{\gamma_{\Delta}} | \xi)) \hat{\lambda}(d\hat{\gamma}_{\Delta}), \end{aligned} \quad (15)$$

where F is a positive measurable function on $\Gamma(X, S)$ and $\hat{\eta} = (\eta, \xi) \in \Gamma(X, S)$. The normalizing factor $Z_{\Delta}(\hat{\eta})$ makes $\Pi_{\Delta}(d\hat{\gamma} | \hat{\eta})$ a probability measure on $\Gamma(X, S)$.

The family $\Pi := \{\Pi_{\Delta}(d\hat{\gamma} | \hat{\eta})\}_{\Delta \in \mathcal{B}_c(X), \hat{\eta} \in \Gamma(X, S)}$ constitutes a Gibbsian specification on $\Gamma(X, S)$ (in the standard sense, see e.g. [13], [34]). In particular, it satisfies the consistency property

$$\int_{\Gamma(X, S)} \Pi_{\Delta_1}(B | \hat{\gamma}) \Pi_{\Delta_2}(d\hat{\gamma} | \hat{\eta}) = \Pi_{\Delta_2}(B | \hat{\eta}), \quad (16)$$

which holds for any $B \in \mathcal{B}(\Gamma(X, S))$, $\hat{\eta} \in \Gamma(X, S)$ and $\Delta_1, \Delta_2 \in \mathcal{B}_c(X)$ such that $\Delta_1 \subset \Delta_2$ (and thus $\hat{\Delta}_1 \subset \hat{\Delta}_2$).

Let ν be a probability measure on $\Gamma(X, S)$. We say that ν is a Gibbs measure associated with the specification Π if it satisfies the DLR equation

$$\nu(B) = \int_{\Gamma(X, S)} \Pi_{\Delta}(B | \hat{\gamma}) \nu(d\hat{\gamma}) \quad (17)$$

for all $B \in \mathcal{B}(\Gamma(X, S))$ and $\Delta \in \mathcal{B}_c(X)$. We denote by \mathcal{G} the set of all such measures.

In what follows, we introduce certain conditions on the interaction potentials Φ, J, W, V , which will guarantee that $\mathcal{G}^t \neq \emptyset$. For that, we consider a partition of X by ‘elementary’ volumes. Denote by Q_k the cube in X with side length 1, centred at point k , $k = (k^{(1)}, \dots, k^{(d)}) \in \mathbb{Z}^d$, that is,

$$Q_k := \{x = (x^{(1)}, \dots, x^{(d)}) \in X : x^{(i)} \in [k^{(i)} - 1/2, k^{(i)} + 1/2)\}. \quad (18)$$

Assumptions on the interaction potentials are as follows.

(A1) Finite Range: $\Phi(x, y) = 0$, $J_{xy} = 0$ if $|x - y| \geq R$, for some $R > 0$.

(A2) Local strong super stability of H : $\exists P > 2$ such that

$$H(\gamma_k) \geq A_{\Phi} N(\gamma_k)^P - B_{\Phi} N(\gamma_k), \quad \gamma_k \in \Gamma(Q_k), \quad (19)$$

for any $k \in \mathbb{Z}^d$ and some constants $A_\Phi > 0$, $B_\Phi \geq 0$ (which may depend on k). Observe that (19) is equivalent to the following (global) strong super stability condition: $\exists A'_\Phi > 0, B'_\Phi \geq 0$ such that

$$H(\gamma) \geq A'_\Phi \sum_{k \in \mathbb{Z}^d} N(\gamma_k)^P - B'_\Phi N(\gamma), \quad \gamma_k = \gamma_{Q_k}, \quad (20)$$

for any $\gamma \in \Gamma_0(X)$.

(A3) Polynomial bound on W , that is, $\exists r > 0$ and $C_W \in \mathbb{R}$ such that

$$|W(u, v)| \leq |u|^r + |v|^r + C_W, \quad u, v \in X.$$

(A4) Polynomial lower bound on V , that is, $\exists q_V \in \mathbb{N}$ and $a_V, b_V \geq 0$ such that

$$V(s) \geq a_V |s|^{q_V} - b_V, \quad s \in S. \quad (21)$$

(A5) We assume that P , q_V and r satisfy the relations

$$(P - 2)(q_V/r - 1) > 1. \quad (22)$$

Remark 1 *One of the best-understood examples of the strong super stable interaction is given by the potential satisfying the bound $\Phi(x, y) \geq c|x - y|^{-d(1+\epsilon)}$ as $|x - y| \rightarrow 0$, in which case $P = 2 + \epsilon$. For a detailed study and historical comments see [36] and also [21, Remark 4.1.].*

It is obvious that $g(S) < \infty$ under Condition (21). Without loss of generality we assume that g is a probability measure.

2.3 Notations

Throughout the paper, we will use following notations (for $k \in \mathbb{Z}^d$):

$$\Gamma_k := \Gamma(Q_k); \quad \gamma_k := \gamma_{Q_k}$$

$$\widehat{\Gamma}_k := \Gamma(Q_k, S); \quad \widehat{\gamma}_k := \widehat{\gamma}_{Q_k \times S}$$

$\partial k := \{j : \text{dist}(Q_k, Q_j) \leq R\}$, where ‘dist’ is the Euclidean distance between two sets

$N_0 := N(\partial k)$; obviously, it is independent of $k \in \mathbb{Z}^d$ and finite

$$K_\Delta := \{k \in \mathbb{Z}^d : \text{dist}(Q_k, \Delta) \leq R\}, \quad \Delta \in \mathcal{B}_c(X)$$

$$\Delta_R := \{x \in X : \text{dist}(x, \Delta) \leq R\}$$

$$\partial \Delta_R := \Delta_R \setminus \Delta = \Delta^c \cap \Delta_R \in \mathcal{B}_c(X)$$

2.4 Main results

Let us fix $p, q \in \mathbb{N}$ such that $p < P$, $q < q_V$ and

$$(p-2)(qr^{-1}-1) \geq 1. \quad (23)$$

Observe that such p and q exist because of condition (22). Define functions $F : \Gamma_0(X, S) \rightarrow \mathbb{R}$ and $F_\alpha : \Gamma(X, S) \rightarrow \mathbb{R}$ by formulae

$$F(\hat{\gamma}) = N(\gamma)^p + \sum_{x \in \gamma} |\sigma_x|^q, \quad \hat{\gamma} = (\gamma, \sigma), \quad (24)$$

and

$$F_\alpha(\hat{\gamma}) = \sup_{k \in \mathbb{Z}^d} e^{-\alpha|k|} F(\hat{\gamma}_k),$$

respectively. Introduce the space of tempered configurations

$$\Gamma^t(X, S) := \{\hat{\gamma} \in \Gamma(X, S) : F_\alpha(\hat{\gamma}) < \infty \text{ for any } \alpha > 0\} \quad (25)$$

and the set \mathcal{G}^t of Gibbs measures that are supported by $\Gamma^t(X, S)$ (*tempered Gibbs measures*), i.e.

$$\mathcal{G}^t := \{\mu \in \mathcal{G} : \mu(\Gamma^t(X, S)) = 1\}. \quad (26)$$

The following three theorems summarize the main results of this paper.

Theorem 2 (*Existence*)

- (i) *The set \mathcal{G}^t is not empty.*
- (ii) *Any $\mu \in \mathcal{G}^t$ satisfies the estimate*

$$\sup_{k \in \mathbb{Z}^d} \int_{\Gamma(X, S)} \exp\{aF(\hat{\gamma}_k)\} \mu(d\hat{\gamma}) < \infty \quad (27)$$

for all $a \in \mathbb{R}$.

Let $\|\cdot\|_\infty$ denote the usual sup norm .

Theorem 3 (*Uniqueness*) *For any fixed $\beta_0 > 0$, one can find constants $\mathcal{J}_0 = \mathcal{J}_0(\beta_0)$ and $z_0 = z_0(\beta_0)$ such that \mathcal{G}^t is a singleton at all values of $\beta < \beta_0$, $\|J\|_\infty \leq \mathcal{J}_0$ and $z \leq z_0$.*

A result that seems to be completely new for this type of systems is the decay of correlations of the Gibbs measures. Consider bounded functions $G_1, G_2 : \Gamma(X, S) \rightarrow \mathbb{R}_+$, such that G_1 is $\mathcal{B}_{Q_{k_1}}(\Gamma(X, S))$ -measurable and G_2 is $\mathcal{B}_{Q_{k_2}}(\Gamma(X, S))$ -measurable, for some $k_1, k_2 \in \mathbb{Z}^d$. Let $\lceil x \rceil$ denote the smallest integer not less than the real number x .

Set

$$\text{Cov}_\mu(G_1, G_2) := \mu(G_1 G_2) - \mu(G_1)\mu(G_2).$$

Theorem 4 (*Decay of Correlations*) *In the conditions of the above theorem, let μ be the unique tempered Gibbs measure. Then, there exist constants \mathfrak{c} and \mathfrak{a} such that*

$$|\text{Cov}_\mu(G_1, G_2)| \leq \mathfrak{c} \|G_1\|_\infty \|G_2\|_\infty \exp\{-\mathfrak{a} \lceil |k_1 - k_2|/R \rceil\}.$$

Remark 5 *The result is simply a consequence of [5, Theorem 2.7], where a precise form for the constants \mathfrak{c} and \mathfrak{a} is given. Hence, the proof will be omitted.*

Remark 6 *In [1, 19, 23, 29], a theory of Gibbs measures on marked configuration spaces that satisfy Ruelle's stability, respectively superstability conditions has been developed. To this end, one has to require the following bounds on the energy:*

$$\begin{aligned} \widehat{H}(\widehat{\gamma}) &\geq A_1 |\gamma| - B_1 \quad \text{resp.} \\ \widehat{H}(\widehat{\gamma}) &\geq A_2 \sum_{k \in \mathbb{Z}^d} |\gamma_k|^2 - B_2 |\gamma|, \quad \widehat{\gamma} \in \Gamma_0(X, S), \end{aligned} \tag{28}$$

with some $A_1, B_1, A_2, B_2 > 0$. Obviously, this is impossible in the case of unbounded marks $\sigma_x \in \mathbb{R}^d$ and the interactions like in (8)-(9). However, taking the Lyapunov functional $F(\widehat{\gamma}_k)$, cf. (24), instead of the squared counting map $|\gamma_k|^2$ in (28), we can develop an analogue of Ruelle's superstability estimates and construct the corresponding Gibbs states ν satisfying the regularity condition

$$\sup_{K \in \mathbb{N}} \left\{ K^{-d} \sum_{|k| \leq K} F(\widehat{\gamma}_k) \right\} := C(\widehat{\gamma}) < \infty, \quad \forall \widehat{\gamma} \in \Gamma(X, S) \pmod{\nu}.$$

As for the uniqueness problem for such Gibbs states, one has to develop a harmonic analysis on the marked configuration spaces and a theory of the Kirkwood-Salsburg equations for the corresponding correlation functions. So far, this was done via cluster expansions in [23] under condition (28) which, as already mentioned above, does not cover our model. However, these issues are beyond the scope of the present work.

The proofs of the above results will be given in Sections 4 and 5, respectively.

3 Exponential moment estimate

The aim of this section is to prove a uniform estimate of exponential moments of specification kernels, which will be used in the proofs of Theorems 2 and 3. For a subset $\mathcal{K} \subset \mathbb{Z}^d$, consider the union of elementary cubes $Q_{\mathcal{K}} := \bigcup_{k \in \mathcal{K}} Q_k$ (cf. (18)) and the corresponding set $\widehat{Q}_{\mathcal{K}} = Q_{\mathcal{K}} \times S$.

Theorem 7 *For any $a \in \mathbb{R}$ and any fixed $\beta > 0$ there exists a constant $\Psi = \Psi(a) < \infty$ such that for all $\widehat{\zeta} \in \Gamma^t(X, S)$, $\mathcal{K} \subseteq \mathbb{Z}^d$ and $k \in \mathcal{K}$, the following estimate holds:*

$$\limsup_{\mathcal{K} \nearrow \mathbb{Z}^d} \int_{\Gamma(X, S)} \exp \{aF(\widehat{\gamma}_k)\} \Pi_{\widehat{Q}_{\mathcal{K}}} \left(d\widehat{\gamma} \middle| \widehat{\zeta} \right) \leq \Psi. \quad (29)$$

In order to prove the theorem, we need some preparations. Observe first that Condition (20) immediately implies the following lower bound:

$$\inf_{x \neq y} \Phi(x, y) \geq 2(2^{P-1}A'_{\Phi} - B'_{\Phi}). \quad (30)$$

Thus there exists $M \geq 0$ such that

$$\inf_{x \neq y} \Phi(x, y) \geq -M. \quad (31)$$

We start with the formulation of two auxiliary results, which will be proved in Appendix.

Lemma 8 *For any $\gamma_k \in \Gamma_k$, $k \in \mathbb{Z}^d$, and $\eta \in \Gamma(X)$ we have*

$$-H_{Q_k}(\gamma_k | \eta) \leq -A_{\Phi}N(\gamma_k)^P + \frac{MN_0}{2}N(\gamma_k)^2 + B_{\Phi}N(\gamma_k) + \frac{M}{2} \sum_{j \in \partial k} N(\eta_j)^2. \quad (32)$$

Remark 9 *Similarly, inequalities (20) and (31) imply that for any $\Delta \in \mathcal{B}_c(\Gamma)$ we have*

$$\begin{aligned} -H_{\Delta}(\gamma_{\Delta} | \eta) &= -H(\gamma_{\Delta}) - \sum_{x \in \gamma_{\Delta}} \sum_{y \in \eta_{\partial \Delta_R}} \Phi(x, y) \\ &\leq -A'_{\Phi} \sum_{k \in K_{\Delta}} N(\gamma_k)^P + B'_{\Phi}N(\gamma_{\Delta}) + MN(\gamma_{\Delta})N(\eta_{\partial \Delta_R}) \end{aligned} \quad (33)$$

so that

$$-H_{\Delta}(\gamma_{\Delta} | \eta) \leq B'_{\Phi}N(\gamma_{\Delta}) + MN(\gamma_{\Delta})N(\eta_{\partial \Delta_R}). \quad (34)$$

Lemma 10 *For any $\varepsilon > 0$, the conditional spin energy function $E_{\gamma_k, \eta}(\sigma_k | \xi)$ satisfies the following estimate:*

$$|E_{\gamma_k, \eta}(\sigma_k | \xi)| \leq \|J\|_\infty \left[C_1 N(\gamma_k)^{2+\varepsilon-1} + C_2 \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon-1} + C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} + C_4 \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)} \right],$$

where C_1, \dots, C_4 are some positive constants (depending on N_0, C_W and ε) and by $\|J\|_\infty$ we denoted the sup norm of J .

Remark 11 *For any $\Delta \in \mathcal{B}_c(\Gamma)$ we have (similarly to (53)) the inequality*

$$|E_{\gamma_\Delta, \eta}(\sigma_\Delta | \xi)| \leq \|J\|_\infty \left[(N(\eta_{\partial \Delta_R}) + 2N(\gamma_\Delta)) \sum_{x \in \gamma_\Delta} |\sigma_x|^r + N(\gamma_\Delta) \sum_{y \in \eta_{\partial \Delta_R}} |\xi_y|^r + (N(\gamma_\Delta)^2 + N(\gamma_\Delta)N(\eta_{\partial \Delta_R})) C_W \right]. \quad (35)$$

Lemma 12 *The partition function Z_Δ satisfies the estimate*

$$1 \leq Z_\Delta(\hat{\eta}) < \infty \quad (36)$$

for all $\Delta \in \mathcal{B}_c(X)$ and $\hat{\eta} \in \Gamma(X, S)$.

Proof. The first inequality follows from the definition of the Lebesgue-Poisson measure $\hat{\lambda}$ and the fact that $H_\Delta(\gamma_\Delta | \eta) = E_{\gamma_\Delta \cup \eta_{\Delta^c}}(\sigma_{\gamma_\Delta} | \xi) = 0$ provided $\Delta = \emptyset$. The second inequality follows from estimates (33) and (35). \square

Proof of Theorem 7. The proof is technical and will be split into two steps.

Step 1. One-point estimate.

Let us fix $k \in \mathbb{Z}^d$ and introduce the notation

$$\begin{aligned} \mathcal{I}_k(\hat{\eta}) &:= \int_{\hat{\Gamma}_k} \exp\{aF(\hat{\gamma}_k)\} \Pi_{\hat{Q}_k}(d\hat{\gamma}_k | \hat{\eta}) \\ &= \int_{\Gamma_k} \int_{S^\gamma} Z_k^{-1}(\hat{\eta}) \exp \left\{ (aN(\gamma_k)^p - \beta H_{Q_k}(\gamma_k | \eta)) + a \sum_{x \in \gamma_k} |\sigma_x|^q - \beta E_{\gamma_k, \eta}(\sigma_k | \xi) \right\} \\ &\quad \bigotimes_{x \in \gamma_k} g(d\sigma_x) \lambda(d\gamma_k), \end{aligned}$$

where $\widehat{\eta} = (\eta, \xi)$, $\widehat{\gamma} = (\gamma, \sigma)$. Observe that

$$\begin{aligned} \int_{S^\gamma} \exp \left(b_1 \sum_{x \in \gamma_k} |\sigma_x|^{b_2} \right) \bigotimes_{x \in \gamma_k} g(d\sigma_x) &= \left(\int_S \exp \left(b_1 |s|^{b_2} \right) g(ds) \right)^{N(\gamma_k)} \\ &= \exp \left(C_{b_1, b_2} N(\gamma_k) \right), \end{aligned}$$

for any $b_1 \in \mathbb{R}$ and $b_2 < q_V$, where

$$C_{b_1, b_2} = \ln \int_S \exp \left(b_1 |s|^{b_2} \right) g(ds) < \infty.$$

Taking into account that $Z_{Q_k}(\widehat{\eta}) \geq 1$ (cf. (36)) we see that

$$\begin{aligned} \mathcal{I}_k(\widehat{\eta}) &= \mathcal{I}_k(\eta, \xi) \\ &\leq \int_{\Gamma_k} \int_{S^\gamma} \exp \left\{ (aN(\gamma_k)^p - \beta H_{Q_k}(\gamma_k | \eta)) + a \sum_{x \in \gamma_k} |\sigma_x|^q - \beta E_{\gamma_k, \eta}(\sigma_k | \xi) \right\} \\ &\quad \bigotimes_{x \in \gamma_k} dg(\sigma_x) \lambda(d\gamma_k), \quad (37) \end{aligned}$$

which in turn implies (by Lemma 10) the inequality

$$\begin{aligned} &\mathcal{I}_k(\eta, \xi) \\ &\leq \int_{\Gamma_k} \exp \left\{ (aN(\gamma_k)^p - \beta H_{Q_k}(\gamma_k | \eta)) + \beta \|J\|_\infty C_1 N(\gamma_k)^{2+\varepsilon^{-1}} + C_{b_1, b_2} N(\gamma_k) \right\} \lambda(d\gamma_k) \\ &\quad \times \exp \left\{ \beta \|J\|_\infty \left[C_2 \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon^{-1}} + C_4 \sum_{j \in \partial k} \sum_{y \in \eta_j} |\xi_y|^{r(1+\varepsilon)} \right] \right\}, \quad (38) \end{aligned}$$

where $b_1 = \beta \|J\|_\infty C_3 + a$, $b_2 = \max \{r(1+\varepsilon), q\}$. Using estimate (32) we finally obtain

$$\begin{aligned} \mathcal{I}_k(\eta, \xi) &\leq \int_{\Gamma_k} \exp \left(-\beta A_\Phi N(\gamma_k)^P + \mathcal{P}(N(\gamma_k)) \right) \lambda(d\gamma_k) \\ &\times \exp \left\{ \frac{\beta M}{2} \sum_{j \in \partial k} N(\eta_j)^2 + \beta \|J\|_\infty \left[C_2 \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon^{-1}} + C_4 \sum_{j \in \partial k} \sum_{y \in \eta_j} |\xi_y|^{r(1+\varepsilon)} \right] \right\}, \end{aligned}$$

where $\mathcal{P}(N(\gamma_k)) := aN(\gamma_k)^p + \frac{\beta M N_0}{2} N(\gamma_k)^2 + \beta B_\Phi N(\gamma_k) + \beta \|J\|_\infty C_1 N(\gamma_k)^{2+\varepsilon^{-1}} + C_{b_1, b_2} N(\gamma_k)$, so that

$$\mathcal{I}_k(\eta, \xi) \leq e^{C_0} \exp \left\{ \sum_{j \in \partial k} \left[C_5 N(\eta_j)^{2+\varepsilon^{-1}} + \beta \|J\|_\infty C_4 \sum_{y \in \eta_j} |\xi_y|^{r(1+\varepsilon)} \right] \right\} \quad (39)$$

with $C_5 = \beta \|J\|_\infty C_2 + \frac{\beta M}{2}$ and

$$C_0 = C_0(a, \beta) = \ln \int_{\Gamma_k} \exp(-\beta A_\Phi N(\gamma_k)^P + \mathcal{P}(N(\gamma_k))) \lambda(d\gamma_k).$$

Observe that $C_0 < \infty$ because $P > \text{degree } \mathcal{P} = \max(2 + \varepsilon^{-1}, p)$ and $A_\Phi > 0$.

Step 2. Volume estimate.

Introduce the notation $n_k(\mathcal{K}, \hat{\zeta}) := \ln \int_{\Gamma(X, S)} \exp(aF(\hat{\eta}_k)) \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta} | \hat{\zeta})$.

An application of equation (16) shows that

$$n_k(\mathcal{K}, \hat{\zeta}) = \ln \int_{\Gamma(X, S)} \mathcal{I}_k(\hat{\eta}) \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta} | \hat{\zeta}).$$

By inequality (39) we have

$$\begin{aligned} n_k(\mathcal{K}, \hat{\zeta}) &\leq C_0 \\ &+ \ln \int_{\Gamma(X, S)} \exp \left\{ \sum_{j \in \partial k} \left[C_5 N(\eta_j)^{2+\varepsilon^{-1}} + \beta \|J\|_\infty C_4 \sum_{y \in \eta_j} |\xi_y|^{r(1+\varepsilon)} \right] \right\} \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta}, \hat{\zeta}) \\ &\leq C_0 + \ln \int_{\Gamma(X, S)} \exp \left\{ \sum_{j \in \partial k} \left[C_5 N(\eta_j)^p + \beta \|J\|_\infty C_4 \sum_{y \in \eta_j} |\xi_y|^q \right] \right\} \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta}, \hat{\zeta}) \end{aligned}$$

for any ε such that $2 + \varepsilon^{-1} \leq p$ and $r(1 + \varepsilon) \leq q$, that is, $\varepsilon \in [(p - 2)^{-1}, r^{-1}q - 1]$. Observe that $(p - 2)^{-1} \leq r^{-1}q - 1$ because of condition (23). Then

$$n_k(\mathcal{K}, \hat{\zeta}) \leq C_0 + \ln \int_{\Gamma(X, S)} \exp \left\{ a \sum_{j \in \partial k} D \left[N(\eta_j)^p + \sum_{y \in \eta_j} |\xi_y|^q \right] \right\} \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta}, \hat{\zeta}),$$

where $D = \frac{1}{a} \max \{C_5, \beta \|J\|_\infty C_4\}$, which implies that

$$n_k(\mathcal{K}, \hat{\zeta}) \leq C_0 + \ln \int_{\Gamma(X, S)} \prod_{j \in \partial k} (\exp\{aF(\hat{\eta}_j)\})^D \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta}, \hat{\zeta}).$$

Observe that the constants C_4 and C_5 are independent of a and assume without loss of generality that $a \geq \max \{C_5, \beta \|J\|_\infty C_4\} N_0$. Then $DN_0 \leq 1$, and we can apply the multiple Hölder inequality, which yields

$$\int_{\Gamma(X, S)} \prod_{j \in \partial k} (\exp\{aF(\hat{\eta}_j)\})^D \hat{\Pi}_\mathcal{K}(d\hat{\eta}, \hat{\zeta}) \leq \prod_{j \in \partial k} \left(\int_{\Gamma(X, S)} \exp\{aF(\hat{\eta}_j)\} \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\eta}, \hat{\zeta}) \right)^D.$$

Therefore

$$\begin{aligned}
n_k(\mathcal{K}, \widehat{\zeta}) &\leq C_0 + D \sum_{j \in \partial k} n_j(\mathcal{K}, \widehat{\zeta}) \\
&= C_0 + aD \sum_{\substack{j \in \partial k \\ j \notin \mathcal{K}}} F(\widehat{\zeta}_j) + D \sum_{\substack{j \in \partial k \\ j \in \mathcal{K}}} n_j(\mathcal{K}, \widehat{\zeta}) \quad (40)
\end{aligned}$$

Fix arbitrary $k_0 \in \mathcal{K}$ and $\alpha > 0$ such that $e^{\alpha\vartheta}DN_0 < 1$, where $\vartheta = \sup_{k \in \mathbb{Z}^d} \max_{j \in \partial k} |j - k|$. Multiplying both sides of inequality (40) by $e^{-\alpha|k_0 - k|}$ and taking into account that

$$-|k_0 - k| \leq |j - k| - |j - k_0| \leq \vartheta - |j - k_0|,$$

we obtain the estimate

$$\begin{aligned}
n_k(\mathcal{K}, \widehat{\zeta})e^{-\alpha|k_0 - k|} &\leq C_0e^{-\alpha|k_0 - k|} + e^{\alpha\vartheta}aD \sum_{\substack{j \in \partial k \\ j \notin \mathcal{K}}} F(\widehat{\zeta}_j)e^{-\alpha|j - k_0|} \\
&\quad + e^{\alpha\vartheta}D \sum_{\substack{j \in \partial k \\ j \in \mathcal{K}}} n_j(\mathcal{K}, \widehat{\zeta})e^{-\alpha|j - k_0|}. \quad (41)
\end{aligned}$$

Observe that

$$\begin{aligned}
\sup_{\substack{k \in \mathcal{K} \\ j \in \partial k \\ j \in \mathcal{K}}} n_j(\mathcal{K}, \widehat{\zeta})e^{-\alpha|j - k_0|} &\leq N(\partial k) \sup_{k \in \mathcal{K}} \left(n_k(\mathcal{K}, \widehat{\zeta})e^{-\alpha|k_0 - k|} \right) \\
&\leq N_0 \sup_{k \in \mathcal{K}} \left(n_k(\mathcal{K}, \widehat{\zeta})e^{-\alpha|k_0 - k|} \right).
\end{aligned}$$

Applying supremum to both sides of inequality (41) we can see that

$$\begin{aligned}
\sup_{k \in \mathcal{K}} \left(n_k(\mathcal{K}, \widehat{\zeta})e^{-\alpha|k_0 - k|} \right) &\leq C_0 + e^{\alpha\vartheta}aD \sum_{j \notin \mathcal{K}} F(\widehat{\zeta}_j)e^{-\alpha|k_0 - j|} \\
&\quad + e^{\alpha\vartheta}DN_0 \sup_{k \in \mathcal{K}} \left(n_k(\mathcal{K}, \widehat{\zeta})e^{-\alpha|k_0 - k|} \right),
\end{aligned}$$

so that

$$\begin{aligned}
(1 - e^{\alpha\vartheta}DN_0) \sup_{k \in \mathcal{K}} \left(n_k(\mathcal{K}, \widehat{\eta})e^{-\alpha|k_0 - k|} \right) &\leq C_0 + e^{\alpha\vartheta}aD \sum_{j \notin \mathcal{K}} e^{-\alpha|j - k_0|} F(\widehat{\eta}_j) \\
&\leq C_0 + e^{\alpha(\vartheta + k_0)}aDF_\alpha \left(\widehat{\eta}_{\widehat{Q}_{\mathcal{K}}} \right).
\end{aligned}$$

Thus

$$\begin{aligned} n_{k_0}(\mathcal{K}, \hat{\eta}) &\leq \sup_{k \in \mathcal{K}} (n_k(\mathcal{K}, \hat{\eta}) e^{-\alpha|k_0-k|}) \\ &\leq e^{\alpha\vartheta} (1 - e^{\alpha\vartheta} DN_0)^{-1} \left(C_0 + e^{\alpha(\vartheta+k_0)} aDF_\alpha \left(\hat{\eta}_{\hat{Q}_\mathcal{K}^c} \right) \right), \end{aligned} \quad (42)$$

which implies that

$$\limsup_{\mathcal{K} \nearrow \mathbb{Z}^d} n_{k_0}(\mathcal{K}, \hat{\zeta}) \leq e^{\alpha\vartheta} (1 - e^{\alpha\vartheta} DN_0)^{-1} C_0,$$

since

$$F_\alpha \left(\hat{\eta}_{\hat{Q}_\mathcal{K}^c} \right) \rightarrow 0, \quad \mathcal{K} \nearrow \mathbb{Z}^d.$$

Passage to the limit as $\alpha \rightarrow 0$ shows that

$$\limsup_{\mathcal{K} \nearrow \mathbb{Z}^d} n_{k_0}(\mathcal{K}, \hat{\zeta}) \leq (1 - DN_0)^{-1} C_0(a) =: \Psi(a),$$

which completes the proof. \square

Corollary 13 *For any cubic domain $\hat{\Delta}$ and $N \in \mathbb{N}$, there exists $\mathfrak{C}(\hat{\Delta}, N) < \infty$ such that*

$$\limsup_{\mathcal{K} \nearrow \mathbb{Z}^d} \int_{\Gamma(X, S)} F^N(\hat{\gamma}_{\hat{\Delta}}) \Pi_{\hat{Q}_\mathcal{K}}(d\hat{\gamma}|\hat{\zeta}) \leq \mathfrak{C}(\hat{\Delta}, N) < \infty,$$

where $\mathfrak{C}(\hat{\Delta}, N)$ can be chosen uniformly for all $\hat{\zeta} \in \Gamma^t(X, S)$.

4 Existence of Gibbs measures

In this section, we use the estimates obtained in Section 3 in order to prove that, for any $\hat{\eta} \in \Gamma^t(X, S)$, the family of Gibbsian specifications $\{\Pi_\Delta(d\hat{\gamma}|\hat{\eta})\}_{\Delta \in \mathcal{B}_c(X)}$ contains a cluster point.

We define the set

$$\hat{\Gamma}_T := \{\hat{\gamma} \in \Gamma(X, S) : F(\hat{\gamma}_{\Delta_R}) \leq T\}, \quad T > 0.$$

Observe that for any set $\Lambda \in \mathcal{B}_c(X)$ there exists a constant c_Λ such that

$$N(\hat{\gamma}_\Lambda) \leq c_\Lambda T, \quad \hat{\gamma} \in \hat{\Gamma}_T, \quad T > 0. \quad (43)$$

Definition 14 We say that a family of probability measures $\{\mu_m\}_{m \in \mathbb{N}}$ on $\Gamma(X, S)$ is locally equicontinuous (LEC) if for any $\Delta \in \mathcal{B}_c(X)$ and any sequence $\{B_n\}_{n \in \mathbb{N}} \in \mathcal{B}_\Delta(\Gamma(X, S))$, $B_n \downarrow \emptyset$, $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \limsup_{m \in \mathbb{N}} \mu_m(B_n) = 0. \quad (44)$$

We equip the space $\mathcal{P}(\Gamma(X, S))$ of probability measures on $\Gamma(X, S)$ with the following local set convergence:

$$\mu_n \xrightarrow{loc} \mu \text{ iff } \mu_n(B) \rightarrow \mu(B), \quad n \rightarrow \infty, \quad B \in \mathcal{B}_{loc}(\Gamma(X, S)).$$

Observe that the local set convergence is equivalent to convergence in the space $[0, 1]^{\mathcal{F}_0}$, where $\mathcal{F}_0 := \mathcal{B}_{loc}(\Gamma(X, S))$.

Theorem 15 (cf. [13, Prop. 4.9]) Any LEC family of probability measures $\{\mu_N\}_{N \in \mathbb{N}}$ on $\Gamma(X, S)$ has a cluster point, which is a probability measure on $\Gamma(X, S)$.

Proof. It is straightforward that the family $\{\mu_N\}_{N \in \mathbb{N}}$ contains a cluster point μ as an element of the compact space $[0, 1]^{\mathcal{F}_0}$, and μ is an additive function on \mathcal{F}_0 . The LEC property (44) implies that $\mu_\Lambda := p_\Lambda^* \mu$ is σ -additive on each $\mathcal{F}_\Lambda := \mathcal{B}_\Lambda(\Gamma(X, S))$. Thus $\{\mu_\Lambda\}_{\Lambda \in \mathcal{B}_c(X)}$ forms a consistent (w.r.t. projective maps (7)) family of measures and by the corresponding version of the Kolmogorov theorem (see [32, Theorem V.3.2]) generates a probability measure on $(\Gamma(X, S), \mathcal{B}(\Gamma(X, S)))$, (which obviously coincides with μ). \square

Corollary 16 There exists a subsequence $\{\mu_{m_k}\}_{k \in \mathbb{N}}$ such that $\mu_{m_k} \xrightarrow{loc} \mu$, $k \rightarrow \infty$.

Let $\{\Lambda_m\}_{m \in \mathbb{N}} \subset \mathcal{B}_c(X)$ be any increasing sequence of sets such that $\Lambda_m \nearrow X$, $m \rightarrow \infty$, and introduce the notation $\Pi_m(d\hat{\gamma} | \hat{\zeta}) := \Pi_{\Lambda_m}(d\hat{\gamma} | \hat{\zeta})$.

Theorem 17 For any $\hat{\zeta} \in \Gamma^t(X, S)$ the family $\{\Pi_m\}_{m \in \mathbb{N}}$ is LEC.

Proof. Fix $\Delta \in \mathcal{B}_c(X)$ and $\{B_n\}_{n \in \mathbb{N}}$ as in Definition 14. It is sufficient to prove that $\forall \varepsilon > 0$ there exists m_0 and n_0 such that

$$\Pi_m(B_n | \hat{\zeta}) \leq \varepsilon$$

for any $m \geq m_0$ and $n \geq n_0$.

First, we will fix $T > 0$ and estimate the corresponding measures of the sets $B_n \cap \widehat{\Gamma}_T$ and $B_n \cap \left(\widehat{\Gamma}_T\right)^c$. Using bounds (34) and (35), inequality (43) and obvious estimates $\sum_{x \in \eta_\Delta} |\xi_x|^r \leq cF(\widehat{\eta})$, $\sum_{x \in \gamma_{\Delta^c}} |\sigma_x|^r \leq cF(\widehat{\gamma})$ that hold for some constant $c > 0$, we obtain the inequalities

$$\mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\eta}_\Delta \cup \widehat{\gamma}_{\Delta^c}) \exp \left\{ -H_\Delta(\eta_\Delta | \gamma) \right\} \leq \exp \left\{ B'_\Phi T + MT^2 \right\}$$

and

$$\begin{aligned} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\eta}_\Delta \cup \widehat{\gamma}_{\Delta^c}) \exp \left\{ -E_{\gamma_\Delta, \eta}(\xi_{\gamma_\Delta} | \sigma) \right\} \\ \leq \exp \left\{ \|J\|_\infty \left[3T \sum_{x \in \eta_\Delta} |\xi_x|^r + T \sum_{x \in \gamma_{\Delta^c}} |\sigma_x|^r + 2T^2 C_W \right] \right\} \\ \leq \exp \left\{ \|J\|_\infty [cT^2 (4 + 2C_W)] \right\}. \end{aligned}$$

Thus there exists a constant $a(\Delta, T)$ such that

$$\mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\eta}_\Delta \cup \widehat{\gamma}_{\Delta^c}) \exp \left\{ -\beta H_\Delta(\eta_\Delta | \gamma) - \beta E_{\eta_\Delta, \gamma}(\xi_{\eta_\Delta} | \sigma) \right\} \leq a(\Delta, T) \quad (45)$$

for all $\widehat{\eta}, \widehat{\gamma} \in \Gamma(X, S)$ and $n \in \mathbb{N}$.

According to Chebyshev's inequality applied to measure Π_m on $\Gamma(X, S)$ we have

$$\Pi_m \left(\left\{ \widehat{\gamma} \in \Gamma(X, S) : f(\widehat{\gamma}) \geq T \right\} \middle| \widehat{\zeta} \right) \leq T^{-2} \int_{\Gamma(X, S)} |f(\widehat{\gamma})|^2 \Pi_m \left(d\widehat{\gamma} \middle| \widehat{\zeta} \right)$$

for any $T > 0$ and $f \in L^2(\Gamma(X, S), \Pi_m)$. Setting $f(\widehat{\gamma}) = F(\widehat{\gamma}_{\widehat{\Delta}_R})$ we obtain, cf. Corollary 13.

$$\Pi_m \left(\left(\widehat{\Gamma}_T \right)^c \middle| \widehat{\zeta} \right) \leq \varepsilon \quad (46)$$

for any $\varepsilon > 0$ and T greater than some $T(\varepsilon, \widehat{\zeta})$. Now we see that

$$\begin{aligned} \Pi_m \left(B_n \middle| \widehat{\zeta} \right) &= \Pi_m \left(B_n \cap \left(\widehat{\Gamma}_T \right)^c \middle| \widehat{\zeta} \right) + \Pi_m \left(B_n \cap \widehat{\Gamma}_T \middle| \widehat{\zeta} \right) \\ &\leq \Pi_m \left(\left(\widehat{\Gamma}_T \right)^c \middle| \widehat{\zeta} \right) + \int_{\Gamma(X, S)} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\gamma}_\Delta) \Pi_m \left(d\widehat{\gamma} \middle| \widehat{\zeta} \right). \end{aligned}$$

Observe that there exists m_0 such that $\Lambda_m \supset \Delta$ for $m \geq m_0$. For all such m ,

it follows from (14) and consistency property (16) of the specification Π that

$$\begin{aligned}
& \int_{\Gamma(X,S)} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\gamma}_\Delta) \Pi_m(d\widehat{\gamma} | \widehat{\zeta}) \\
&= \int_{\Gamma(X,S)} \left[\int_{\Gamma(\Delta,S)} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\eta}_\Delta \cup \widehat{\gamma}_{\Delta^c}) \Pi_\Delta(d\widehat{\eta} | \widehat{\gamma}) \right] \Pi_m(d\widehat{\gamma} | \widehat{\zeta}) \\
&= \int_{\Gamma(X,S)} Z_\Delta(\widehat{\gamma})^{-1} \int_{\Gamma(\Delta,S)} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\eta}_\Delta \cup \widehat{\gamma}_{\Delta^c}) \\
&\times \exp(-\beta H_\Delta(\gamma_\Delta | \eta) - \beta E_{\gamma_\Delta \cup \eta_{\Delta^c}}(\sigma_{\gamma_\Delta} | \xi)) \bigotimes_{x \in \gamma_\cdot} g(d\sigma_x) \lambda(d\eta_\cdot) \Pi_m(d\widehat{\gamma} | \widehat{\zeta}).
\end{aligned}$$

Thus by (45) and (36) we obtain

$$\int_{\Gamma(\Delta,S)} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\eta}_\Delta \cup \widehat{\gamma}_{\Delta^c}) \Pi_\Delta(d\widehat{\eta} | \widehat{\gamma}) \leq a(\Delta, T) \widehat{\lambda}(B_n) < \varepsilon$$

for n greater than some $n(\varepsilon, T)$. Hence,

$$\int_{\Gamma(X,S)} \mathbf{1}_{B_n \cap \widehat{\Gamma}_T}(\widehat{\gamma}_\Delta) \Pi_m(d\widehat{\gamma} | \widehat{\zeta}) < \varepsilon.$$

Combining this with estimate (46) we can see that $\forall \varepsilon > 0$ and $m \geq m_0$, $n \geq n_0 = n(\varepsilon/2, T(\varepsilon/2))$ we have

$$\Pi_m(\widehat{B}_n | \widehat{\zeta}) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The proof is complete. \square

Now we are in a position to prove our existence result.

Proof of Theorem 2. It follows from Theorems 15 and 17 that for any $\widehat{\zeta} \in \Gamma^t(X, S)$ the family $\left\{ \Pi_m(d\widehat{\gamma} | \widehat{\zeta}) \right\}_{m \in \mathbb{N}}$ has a cluster point $\mu = \mu(\widehat{\zeta}) \in \mathcal{P}(\Gamma(X, S))$, so that there exists a subsequence $\widehat{\Lambda}_{m_j}$, $j \in \mathbb{N}$, such that

$$\mu(B) = \lim_{j \rightarrow \infty} \Pi_{m_j}(B | \widehat{\zeta}), \tag{47}$$

for any $B \in \mathcal{B}_{loc}(\Gamma(X, S))$. Standard limit transition arguments and the consistency property (16) of the specification Π show that μ satisfies the DLR equation (17) and (27), hence the result follows. \square

5 Uniqueness of Gibbs measures

5.1 Lattice representation of the model

Let $\mathcal{Q} := \Gamma(\overline{Q_0}, S)$, where $\overline{Q_0}$ is the closure of the elementary cube Q_0 centered at the origin. Consider the product space $\mathcal{A} := \mathcal{Q}^{\mathbb{Z}^d} = \prod_{k \in \mathbb{Z}^d} \mathcal{Q}_k$, $\mathcal{Q}_k = \mathcal{Q}$ and equip it with the product topology and the corresponding Borel σ -algebra $\mathcal{B}(\mathcal{A})$. By Section A.5 in [23] and Remark 4.A3 in [13], $(\mathcal{A}, \mathcal{B}(\mathcal{A}))$ is a standard Borel space. For $k \in \mathbb{Z}^d$, let ω_k be the projection of $\bar{\omega}$ onto \mathcal{Q}_k . By construction, $\mathcal{B}(\mathcal{A})$ is generated by cylinder sets $A_{b_1, \dots, b_m}^{k_1, \dots, k_m} := \{\bar{\omega} \in \mathcal{A} : \omega_{k_1} \in b_1, \dots, \omega_{k_m} \in b_m\}$, $b_1, \dots, b_m \in \mathcal{B}(\mathcal{Q})$. Consider also the cylinder sets $C_{b_1, \dots, b_m}^{k_1, \dots, k_m} := \{\hat{\gamma} \in \Gamma(X, S) : \hat{\gamma}_{\overline{Q_{k_l}}} - k_l \in b_l, 1 \leq l \leq m\} \in \mathcal{B}(\Gamma(X, S))$.

Define the map

$$T : \Gamma(X, S) \ni \hat{\gamma} \mapsto T(\hat{\gamma}) = \bar{\omega} \in \mathcal{A}$$

where $\bar{\omega} = (\omega_k)_{k \in \mathbb{Z}^d}$ and $\omega_k = \hat{\gamma} \cap (\overline{Q_k} \times S) - k \in \Gamma(\overline{Q_0}, S)$. We use the notation $\hat{\eta} - a = \{\dots, (x - a, s), \dots\}$ for a marked configuration $\hat{\eta} = \{\dots, (x, s), \dots\} \in \Gamma(X, S)$ and $a \in X$.

Remark 18 *Observe that in our notations \mathcal{Q}_k is the k -th copy of $\mathcal{Q} = \Gamma(\overline{Q_0}, S)$, so that $\mathcal{Q}_k \neq \Gamma(\overline{Q_k}, S)$. These spaces are isomorphic via the translation by k .*

Lemma 19 (i) $T : \Gamma(X, S) \rightarrow \mathcal{A}$ is measurable;

(ii) $T(B) \in \mathcal{B}(\mathcal{A})$ for any $B \in \mathcal{B}(\Gamma(X, S))$.

Proof. (i) Notice that

$$T^{-1} \left(A_{b_1, \dots, b_m}^{k_1, \dots, k_m} \right) = C_{b_1, \dots, b_m}^{k_1, \dots, k_m},$$

which proves the statement.

(ii) For a compact $\Lambda \subset X$ and $\mathfrak{b} \in \mathcal{B}(\Gamma(\Lambda, S))$ consider the cylinder set

$$B_{\Lambda, \mathfrak{b}} := \{\hat{\gamma} \in \Gamma(X, S) : \hat{\gamma}_{\Lambda} \in \mathfrak{b}\} \in \mathcal{B}_0(\Gamma(X, S)). \quad (48)$$

Assume that $\Lambda \subset \bigcup_{i=1}^m Q_{k_i}$. For $B_{\Lambda, \mathfrak{b}}$ defined by (48) we have

$$T(B_{\Lambda, \mathfrak{b}}) = \{\bar{\omega} \in \mathcal{A} : \bigcup_{i=1}^m (\omega_{k_i} + k_i) \in \mathfrak{b}\},$$

which is measurable. \square

Thus, for any $\mu \in \mathcal{P}(\Gamma(X, S))$ we can define its push-forward image $T_*\mu \in \mathcal{P}(\mathcal{A})$, where $\mathcal{P}(\mathcal{A})$ is the set of all probability measures on \mathcal{A} .

Lemma 20 *The map $T_* : \mathcal{P}(\Gamma(X, S)) \rightarrow \mathcal{P}(\mathcal{A})$ is injective.*

Proof. Let $\mu, \nu \in \mathcal{P}(\Gamma(X, S))$ and $\mu \neq \nu$. Then there exists $B \in \mathcal{B}(\Gamma(X, S))$ such that $\mu(B) \neq \nu(B)$. By Lemma 19, $A := T(B) \in \mathcal{B}(\mathcal{A})$. The injectivity of T implies that $T^{-1}(T(B)) = B$. Thus $T_*\mu(A) = \mu(T^{-1}(A)) \neq \nu(T^{-1}(A)) = T_*\nu(A)$, and the statement is proved. \square

Let $\mathcal{A}' := T(\Gamma(X, S))$. Define a family of one-point states $\mathfrak{M} = \{\mathfrak{m}_k(\cdot | \bar{\omega}), k \in \mathbb{Z}^d, \bar{\omega} \in \mathcal{A}'\}$ on \mathcal{A} by the formula

$$\mathfrak{m}_k(b | \bar{\omega}) = \mu_k(b + k | T^{-1}\bar{\omega}), \quad b \in \mathcal{B}(\Gamma(\bar{Q}, S)),$$

where $\mu_k := \mu_{\hat{Q}_k}$ is the local Gibbs state defined by formula (12), and the corresponding one-point specification $\mathfrak{P} = \{\mathfrak{p}_k(\cdot | \bar{\omega}), k \in \mathbb{Z}^d, \bar{\omega} \in \mathcal{A}'\}$, where

$$\mathfrak{p}_k(B | \bar{\omega}) = \Pi_{\hat{Q}_k}(T^{-1}(B) | T^{-1}\bar{\omega}), \quad B \in \mathcal{B}(\mathcal{A}).$$

It is clear that $\mathfrak{m}_k(\cdot | \bar{\omega})$ coincides with the projection of $\mathfrak{p}_k(\cdot | \bar{\omega})$ onto the k -th component of the product space \mathcal{A} . Observe that, by construction, all measures $\mathfrak{p}_k(\cdot | \bar{\omega})$ are concentrated on \mathcal{A}' .

Remark 21 *Consider the set*

$$\mathring{\Gamma}(X, S) := \{\hat{\gamma} \in \Gamma(X, S) \mid \gamma \cap \partial Q_k = \emptyset, \quad \forall k \in \mathbb{Z}^d\} \in \mathcal{B}(\Gamma(X, S)).$$

The space $\mathring{\mathcal{A}}' := T(\mathring{\Gamma}(X, S))$ can easily be identified with the space $\mathring{\mathcal{Q}}^{\mathbb{Z}^d}$, where $\mathring{\mathcal{Q}} := \{\hat{\gamma} \in \Gamma(\bar{Q}, S) : \hat{\gamma} \cap ((\bar{Q} \setminus Q) \times S) = \emptyset\}$. Moreover, from the properties of the marked Lebesgue-Poisson measure we can deduce that $\hat{\lambda}(\mathring{\Gamma}(X, S)) = 1$.

Lemma 22 *For any $k \in \mathbb{Z}^d$ we have the following.*

(i) *Measure $\mathfrak{m}_k(d\bar{\varsigma} | \bar{\omega})$ has the form*

$$\mathfrak{m}_k(d\bar{\varsigma} | \bar{\omega}) = Z^{-1} e^{-H_k(\bar{\varsigma} | \bar{\omega})} T_* \hat{\lambda}_Q(d\varsigma),$$

where $H_k(\bar{\varsigma} | \bar{\omega}) = \hat{H}_{\hat{Q}_k}(\bar{\varsigma} + k | T^{-1}\bar{\omega})$, $\bar{\varsigma} \in \Gamma(\bar{Q}, S)$ and Z is the normalizing factor; moreover, $Z = Z_{Q_k}(T^{-1}\bar{\omega})$ (cf. (13)).

(ii) *Assume that $\bar{\omega}_{\partial k}^1 = \bar{\omega}_{\partial k}^2$, where ∂k is defined in Sec. 2.3. Then $\mathfrak{m}_k(d\bar{\varsigma} | \bar{\omega}^1) = \mathfrak{m}_k(d\bar{\varsigma} | \bar{\omega}^2)$.*

Proof. The statement immediately follows from the definition of measure $\mathbf{m}_k(d\bar{\varsigma}|\bar{\omega})$ and energy function \hat{H}_{Q_k} , cf. (10), and properties of measure μ_{Q_k} . \square

We denote by $\mathcal{M}(\mathfrak{P})$ and the set of probability measures $\varrho \in \mathcal{P}(\mathcal{A})$ which are consistent with the specification \mathfrak{P} , that is, $\varrho(\mathcal{A}') = 1$ and

$$\int_{\mathcal{A}} \mathbf{p}_k(B|\bar{\omega}) \varrho(d\bar{\omega}) = \varrho(B), \quad B \in \mathcal{B}(\mathcal{A}). \quad (49)$$

Lemma 23 *Let $\mu \in \mathcal{G}$. Then $T_*\mu \in \mathcal{M}(\mathfrak{P})$.*

Proof. By construction, $T_*\mu(\mathcal{A}') = 1$. The consistency property (49) follows directly from DLR equation (17). \square

The next statement is crucial for our approach.

Proposition 24 *We have $N(\mathcal{G}) \leq N(\mathcal{M}(\mathfrak{P}))$.*

Proof. Follows directly from Lemmas 20 and 23. \square

Thus, in order to show that \mathcal{G} contains at most one element, it is sufficient to prove that $\mathcal{M}(\mathfrak{P})$ does so.

The next statement is an adaptation of [5, Theorem 2.6] to our setting. This result originally appeared in the work of Dobrushin and Pechersky [12], see also [33, Theorem 4] and [2, Theorem 3], where its application to the case of interactions in continuum is presented. Denote by d_{var} the total variation distance between two measures on \mathcal{Q} , that is,

$$d_{var}(\theta_1, \theta_2) := \sup_{A \in \mathcal{B}(\mathcal{Q})} |\theta_1(A) - \theta_2(A)|$$

and by d_d the discrete distance between two elements of \mathcal{Q} . Fix a measurable non-negative function $h : \mathcal{Q} \rightarrow \mathbb{R}$ and set

$$\mathcal{M}_h(\mathfrak{P}) := \left\{ \varrho \in \mathcal{M}(\mathfrak{P}) : \sup_{k \in \mathbb{Z}^d} \int_{\mathcal{A}} h(\omega_k) \varrho(d\bar{\omega}) < \infty \right\}.$$

Theorem 25 *Assume that the family of one-point local Gibbs states $\{\mathbf{m}_k(d\bar{\varsigma}|\bar{\omega})\}$ satisfies the following conditions:*

- *there exist a big enough constant $\bar{L} > 0$ and sufficiently small constant l such that*

$$d_{var}(\mathbf{m}_k(d\bar{\varsigma}|\bar{\omega}^1), \mathbf{m}_k(d\bar{\varsigma}|\bar{\omega}^2)) < l \sum_{j \in \partial k} d_d(\bar{\omega}_j^1, \bar{\omega}_j^2), \quad (50)$$

for any $k \in \mathbb{Z}^d$ and all boundary conditions $\bar{\omega}^1, \bar{\omega}^2$ satisfying the relation

$$h(\bar{\omega}_j^1), h(\bar{\omega}_j^2) \leq \bar{L}, \quad \text{for any } j \in \partial k; \quad (51)$$

- there exists a sufficiently small constant c such that

$$\int h(\bar{\varsigma}_k) \mathfrak{m}_k(d\bar{\varsigma}|\bar{\omega}) \leq 1 + c \sum_{j \in \partial k} h(\bar{\omega}_j) \quad (52)$$

for any $k \in \mathbb{Z}^d$, and all boundary conditions $\bar{\omega}$.

Then $N(\mathcal{M}_h(\mathfrak{P})) \leq 1$.

Remark 26 • The result [5, Theorem 2.6] is more refined in that precise bounds on the parameters \bar{L}, l and c are given.

- Note also that this result is stated for particle systems in a discrete setting (i.e. lying on a graph), hence the need to define the corresponding lattice system. Moreover, the precise bounds on the parameters depend on the geometrical properties of the graph. Furthermore, \bar{L} is completely determined by the other parameters.

Remark 27 It is important to observe that if $\mu \in \mathcal{G}$, then $T_*\mu$ is also supported by a smaller set of tempered configurations, namely

$$\mathcal{A}^t := \left\{ \bar{\omega} \in \mathcal{A} : |\bar{\omega}|_\alpha := \sup_{k \in \mathbb{Z}^d} \exp\{-\alpha|k|\} h(\bar{\omega}_k) < \infty, \text{ for some } \alpha > 0 \right\}.$$

Moreover, by (27) it will be easy to see that $T_*\mu$ satisfies the exponential moment estimate

$$\sup_{k \in \mathbb{Z}^d} \int \exp\{ah(\bar{\omega}_k)\} T_*\mu(\bar{\omega}) < \infty, \quad a \in \mathbb{R}.$$

Hence $T_*\mu \in \mathcal{M}_h(\mathfrak{P})$.

5.2 Proof of the uniqueness

In this section, we prove sufficient conditions for the uniqueness of tempered Gibbs measures due to small interaction and small activity parameter formulated in Theorem 3. We will use the lattice representation developed in the previous section and Theorem 25.

Proof of Theorem 3. We check conditions of Theorem 25 for the corresponding lattice model through the following two results.

Lemma 28 Let $0 < \beta < \beta_0$. Then one can find a measurable non-negative function $h : \mathcal{Q} \rightarrow \mathbb{R}$ and a small enough $c > 0$ such that for every $k \in \mathbb{Z}^d$ and any boundary condition $\bar{\omega} \in \mathcal{A}$, (52) is satisfied.

For h given by Lemma 28 we can prove now the other condition in Theorem 25.

Lemma 29 *For some big enough $L > 0$ one can find $\mathcal{J}_0 = \mathcal{J}_0(\beta_0)$ and $z_0 = z_0(\beta_0)$ such that at all values $\beta < \beta_0$, $\|J\|_\infty \leq \mathcal{J}_0$ and $z \leq z_0$ (50) holds for any boundary conditions satisfying (51).*

This completes the proof. \square

6 Appendix: proofs of auxiliary results

Proof of Lemma 8. By the definition of conditional energy $H_{Q_k}(\gamma_k | \eta)$ we have

$$\begin{aligned}
-H_{Q_k}(\gamma_k | \eta) &= -H(\gamma_k) - \sum_{x \in \gamma_k} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} \Phi(x, y) \\
&\leq -\left(A_\Phi N(\gamma_k)^P - B_\Phi N(\gamma_k)\right) + MN(\gamma_k) \sum_{j \in \partial k} N(\eta_j) \\
&\leq -\left(A_\Phi N(\gamma_k)^P - B_\Phi N(\gamma_k)\right) + \frac{M}{2} N(\partial k) N(\gamma_k)^2 + \frac{M}{2} \sum_{j \in \partial k} N(\eta_j)^2 \\
&= -A_\Phi N(\gamma_k)^P + \frac{MN_0}{2} N(\gamma_k)^2 + B_\Phi N(\gamma_k) + \frac{M}{2} \sum_{j \in \partial k} N(\eta_j)^2,
\end{aligned}$$

and the proof is complete. \square

Proof of Lemma 10. By definition (11) of conditional energy function $E_{\gamma_k, \eta}(\sigma_k | \xi)$ we have

$$\begin{aligned}
|E_{\gamma_k, \eta}(\sigma_k | \xi)| &\leq \|J\|_\infty \left[\sum_{\{x, y\} \in \gamma_k} |W(\sigma_x, \sigma_y)| + \sum_{x \in \gamma_k} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |W(\sigma_x, \xi_y)| \right] \\
&\leq \|J\|_\infty \left[\sum_{\{x, y\} \in \gamma_k} (|\sigma_x|^r + |\sigma_y|^r + C_W) + \sum_{x \in \gamma_k} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} (|\sigma_x|^r + |\xi_y|^r + C_W) \right] \\
&\leq \|J\|_\infty \left[\left(\sum_{j \in \partial k} N(\eta_j) + 2N(\gamma_k) \right) \sum_{x \in \gamma_k} |\sigma_x|^r + N(\gamma_k) \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^r \right. \\
&\quad \left. + N(\gamma_k) \left(\frac{N(\gamma_k) - 1}{2} + \sum_{j \in \partial k} N(\eta_j) \right) C_W \right]. \quad (53)
\end{aligned}$$

Observe that $\frac{1}{1+\varepsilon} + \frac{1}{1+\varepsilon^{-1}} = 1$. Fix arbitrary $\rho_1, \rho_2 > 0$ and let be ρ'_1, ρ'_2 such that $\frac{1}{\rho_k} + \frac{1}{\rho'_k} = 1$, $k = 1, 2$. In what follows, we will estimate each of the three terms (53) by Holder's inequality.

For the first term we obtain:

$$\begin{aligned}
A_1 &:= \left(\sum_{j \in \partial k} N(\eta_j) + 2N(\gamma_k) \right) \sum_{x \in \gamma_k} |\sigma_x|^r = \sum_{j \in \partial k} N(\eta_j) \sum_{x \in \gamma_k} |\sigma_x|^r + 2N(\gamma_k) \sum_{x \in \gamma_k} |\sigma_x|^r \\
&\leq \frac{1}{1+\varepsilon^{-1}} N(\gamma_k) \sum_{j \in \partial k} N(\eta_j)^{1+\varepsilon^{-1}} + \frac{N(\partial k)}{1+\varepsilon} \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} \\
&\quad + \frac{2}{1+\varepsilon^{-1}} N(\gamma_k)^{2+\varepsilon^{-1}} + \frac{2}{1+\varepsilon} \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} \\
&= \frac{1}{1+\varepsilon^{-1}} N(\gamma_k) \sum_{j \in \partial k} N(\eta_j)^{1+\varepsilon^{-1}} + \frac{2}{1+\varepsilon^{-1}} N(\gamma_k)^{2+\varepsilon^{-1}} + \frac{N(\partial k) + 2}{1+\varepsilon} \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} \\
&\leq \frac{N(\partial k)}{(1+\varepsilon^{-1}) \rho_1} N(\gamma_k)^{\rho_1} + \frac{1}{(1+\varepsilon^{-1}) \rho'_1} \sum_{j \in \partial k} N(\eta_j)^{(1+\varepsilon^{-1}) \rho'_1} \\
&\quad + \frac{2}{1+\varepsilon^{-1}} N(\gamma_k)^{2+\varepsilon^{-1}} + \frac{N(\partial k) + 2}{1+\varepsilon} \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)}.
\end{aligned}$$

The middle term can be estimated as follows:

$$\begin{aligned}
A_2 &:= N(\gamma_k) \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^r = \sum_{\substack{j \in \partial k \\ y \in \eta_j}} N(\gamma_k) |\xi_y|^r \\
&\leq \frac{1}{1+\varepsilon^{-1}} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} N(\gamma_k)^{1+\varepsilon^{-1}} + \frac{1}{1+\varepsilon} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)} \\
&= \frac{1}{1+\varepsilon^{-1}} \sum_{j \in \partial k} N(\eta_j) N(\gamma_k)^{1+\varepsilon^{-1}} + \frac{1}{1+\varepsilon} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)} \\
&\leq \frac{1}{(1+\varepsilon^{-1}) \rho_2} \sum_{j \in \partial k} N(\eta_j)^{\rho_2} + \frac{1}{(1+\varepsilon^{-1}) \rho'_2} \sum_{j \in \partial k} N(\gamma_k)^{(1+\varepsilon^{-1}) \rho'_2} + \frac{1}{1+\varepsilon} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)} \\
&= \frac{1}{(1+\varepsilon^{-1}) \rho_2} \sum_{j \in \partial k} N(\eta_j)^{\rho_2} + \frac{1}{(1+\varepsilon^{-1}) \rho'_2} N(\partial k) N(\gamma_k)^{(1+\varepsilon^{-1}) \rho'_2} + \frac{1}{1+\varepsilon} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)}.
\end{aligned}$$

Finally, for the last term we have the inequality

$$\begin{aligned}
A_3 &:= N(\gamma_k) \left(\frac{N(\gamma_k) - 1}{2} + \sum_{j \in \partial k} N(\eta_j) \right) C_W \\
&= \frac{C_W}{2} (N(\gamma_k)^2 - N(\gamma_k)) + C_W N(\gamma_k) \sum_{j \in \partial k} N(\eta_j) \\
&\leq \frac{C_W}{2} (N(\gamma_k)^2 - N(\gamma_k)) + \frac{C_W}{\rho_3} N(\partial k) N(\gamma_k)^{\rho_3} + \frac{C_W}{\rho'_3} \sum_{j \in \partial k} N(\eta_j)^{\rho'_3}.
\end{aligned}$$

In order to simplify the expressions above, we set

$$\rho_1 = \rho_2 = 2 + \varepsilon^{-1}, \quad \rho_3 = 2.$$

Then

$$\rho'_1(1 + \varepsilon^{-1}) = \rho'_2(1 + \varepsilon^{-1}) = 2 + \varepsilon^{-1}, \quad \rho'_3 = 2.$$

Using these values, we obtain the following inequalities:

$$\begin{aligned}
A_1 &\leq \frac{N(\partial k)}{(1 + \varepsilon^{-1})(2 + \varepsilon^{-1})} N(\gamma_k)^{2+\varepsilon^{-1}} + \frac{1}{2 + \varepsilon^{-1}} \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon^{-1}} \\
&\quad + \frac{2}{1 + \varepsilon^{-1}} N(\gamma_k)^{2+\varepsilon^{-1}} + \frac{N(\partial k) + 2}{1 + \varepsilon} \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)}; \quad (54)
\end{aligned}$$

$$\begin{aligned}
A_2 &\leq \frac{1}{(1 + \varepsilon^{-1})(2 + \varepsilon^{-1})} \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon^{-1}} + \frac{1}{2 + \varepsilon^{-1}} N(\partial k) N(\gamma_k)^{2+\varepsilon^{-1}} \\
&\quad + \frac{1}{1 + \varepsilon} \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)}; \quad (55)
\end{aligned}$$

$$\begin{aligned}
A_3 &\leq \frac{C_W}{2} (N(\gamma_k)^2 - N(\gamma_k)) + \frac{C_W}{2} N(\partial k) N(\gamma_k)^2 + \frac{C_W}{2} \sum_{j \in \partial k} N(\eta_j)^2 \\
&= \frac{C_W}{2} ((1 + N(\partial k)) N(\gamma_k)^2 - N(\gamma_k)) + \frac{C_W}{2} \sum_{j \in \partial k} N(\eta_j)^2. \quad (56)
\end{aligned}$$

Thus we have the estimate

$$\begin{aligned}
|E_{\gamma_k, \eta}(\sigma_k | \xi)| &\leq \|J\|_\infty \left[C_1 N(\gamma_k)^{2+\varepsilon^{-1}} + C_2 \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon^{-1}} + C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} \right. \\
&\quad \left. + C_4 \sum_{\substack{j \in \partial k \\ y \in \eta_j}} |\xi_y|^{r(1+\varepsilon)} \right]
\end{aligned}$$

for constants C_1, \dots, C_4 explicit form of which can be seen directly from inequalities (54)-(56): $C_1 = \frac{N_0}{(1+\varepsilon^{-1})(2+\varepsilon^{-1})} + \frac{N_0}{2+\varepsilon^{-1}} + \frac{C_W}{2}(2+N_0)$, $C_2 = \frac{1}{2+\varepsilon^{-1}} + \frac{1}{(1+\varepsilon^{-1})(2+\varepsilon^{-1})} + \frac{C_W}{2}$, $C_3 = \frac{N_0+2}{1+\varepsilon}$, $C_4 = \frac{1}{1+\varepsilon}$. \square

Proof of Lemma 28.

Let $\bar{F} := F \circ T$. Let us notice that, by a simple change of variables

$$\int C\bar{F}(\bar{\varsigma}_k)\mathbf{m}_k(d\bar{\varsigma}|\bar{\omega}) = \int CF(\hat{\gamma}_k)\mu_k(d\hat{\gamma}_k|T^{-1}\bar{\omega}),$$

for any positive constant $C > 0$.

The application of Jensen's inequality to (39) shows that

$$\begin{aligned} \int_{\hat{\Gamma}_k} F(\hat{\gamma}_k)\mu_k(d\hat{\gamma}_k|\hat{\eta}) &\leq a^{-1}C_0(\beta, a) \\ &+ a^{-1} \sum_{j \in \partial k} \left[C_5 N(\eta_j)^{2+\varepsilon^{-1}} + \beta \|J\|_\infty C_4 \sum_{y \in \eta_j} |\xi_y|^{r(1+\varepsilon)} \right], \end{aligned}$$

which in turn implies the bound

$$\int_{\hat{\Gamma}_k} CF(\hat{\gamma}_k)\mu_k(d\hat{\gamma}_k|\hat{\eta}) \leq 1 + c \sum_{j \in \partial k} CF(\hat{\eta}_j)$$

with constants $C := aC_0^{-1}(\beta, a)$ and $c := a^{-1} \max\{C_5, \beta \|J\|_\infty C_4\}$ for any $a > 0$, so that c can always be made arbitrary small by the appropriate choice of a . Thus, we can choose $h = C\bar{F}$ and hence condition (52) holds. \square

Proof of Lemma 29. Fix $0 < l < 1$ and let $\bar{\omega}^1, \bar{\omega}^2$ be such boundary conditions satisfying (51). Again, by a simple argument of change of variables, it is easy to see that

$$d_{var}(\mathbf{m}_k(d\bar{\varsigma}|\bar{\omega}^1), \mathbf{m}_k(d\bar{\varsigma}|\bar{\omega}^2)) = d_{var}(\mu_k(d\hat{\gamma}_k|T^{-1}\bar{\omega}^1), \mu_k(d\hat{\gamma}_k|T^{-1}\bar{\omega}^2)).$$

Let $\hat{\eta} := T^{-1}\bar{\omega}^1$ and $\hat{\theta} := T^{-1}\bar{\omega}^2$. Condition (51) implies $F(\hat{\eta}) \leq L := C^{-1}\bar{L}$, which in turn means that $N(\eta_k) \leq L$ and $\sum_{x \in \eta_j} |\xi_x|^r \leq L$, $\hat{\eta} = (\eta, \xi)$.

The general formula for computing the total variation distance between two probability measures with densities with respect to a given measure gives the inequality

$$\begin{aligned} \mathfrak{D} &:= d_{var}(\mu_k(d\hat{\gamma}_k|\hat{\eta}), \mu_k(d\hat{\gamma}_k|\hat{\theta})) \\ &\leq \frac{1}{2} \int_{\hat{\Gamma}_k} \left| Z_{\hat{Q}_k}^{-1}(\hat{\eta}) \exp\left\{-\beta \hat{H}_k(\hat{\gamma}_k|\hat{\eta})\right\} - Z_{\hat{Q}_k}^{-1}(\hat{\theta}) \exp\left\{-\beta \hat{H}_k(\hat{\gamma}_k|\hat{\theta})\right\} \right| \hat{\lambda}(d\hat{\gamma}_k). \end{aligned}$$

Multiplying the right-hand side by the expression $Z_{\widehat{Q}_k}(\widehat{\eta})Z_{\widehat{Q}_k}(\widehat{\theta}) \geq 1$ we see by elementary calculation that

$$\mathfrak{D} \leq \min \left(Z_{\widehat{Q}_k}(\widehat{\eta}), Z_{\widehat{Q}_k}(\widehat{\theta}) \right) \int_{\widehat{\Gamma}_k} \left| \exp\{-\beta \widehat{H}_k(\widehat{\gamma}_k|\widehat{\eta})\} - \exp\{-\beta \widehat{H}_k(\widehat{\gamma}_k|\widehat{\theta})\} \right| \widehat{\lambda}(d\widehat{\gamma}_k).$$

For simplicity we let $\widehat{\eta} = \emptyset$ outside \widehat{Q}_j for a $j \in \partial k$ and $\widehat{\theta} = \emptyset$. Then $Z_{\widehat{Q}_k}(\widehat{\theta}) = 1$, so that

$$\begin{aligned} \mathfrak{D} &\leq \int_{\widehat{\Gamma}_k} \left| \exp\{-\beta \widehat{H}_k(\widehat{\gamma}_k|\widehat{\eta})\} - \exp\{-\beta \widehat{H}_k(\widehat{\gamma}_k|\widehat{\theta})\} \right| \widehat{\lambda}(d\widehat{\gamma}_k) \\ &= \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} \exp \left\{ -\beta [H_k(\gamma_k) + E_{\gamma_k}(\sigma_k)] \right\} \times \\ &\quad \times \left| 1 - \exp \left\{ -\beta \left[\sum_{\substack{x \in \gamma_k \\ y \in \eta_j}} (\Phi(x, y) + J_{xy} W(\sigma_x, \xi_y)) \right] \right\} \right| \widehat{\lambda}(d\widehat{\gamma}_k). \end{aligned}$$

Denote $\mathcal{J} := \|J\|_\infty$.

In what follows, we give separate estimates for the factors in the right-hand side of the above inequality. The (A2) assumption given by (19) and Lemma 10 imply that

$$\exp\{-\beta H_k(\gamma_k)\} \leq \exp\{-\beta A_\Phi N(\gamma_k)^P + \beta B_\Phi N(\gamma_k)\}, \quad (57)$$

and

$$\exp\{-\beta E_{\gamma_k}(\sigma_k)\} \leq \exp\{\beta \mathcal{J} C_1 N(\gamma_k)^{2+\varepsilon^{-1}} + \beta \mathcal{J} C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)}\}, \quad (58)$$

respectively.

For the last factor, we use a few computation tricks. Write $\Phi(x, y) = \Phi^+(x, y) - \Phi^-(x, y)$ and $W(\sigma_x, \xi_y) = W^+(\sigma_x, \xi_y) - W^-(\sigma_x, \xi_y)$, where subscripts $^+$ and $^-$ denote the positive and negative parts of the corresponding function. Applying the elementary inequality

$$\left| 1 - \exp \left(a - \sum_{i=1}^n a_i \right) \right| \leq (e^a - 1) + \sum_{i=1}^n |1 - e^{-a_i}|,$$

where $n \in N$ and $a, a_i \geq 0, 1 \leq i \leq n$, we obtain

$$\begin{aligned}
& |1 - \exp\{-\beta[\sum_{\substack{x \in \gamma_k \\ y \in \eta_j}} (\Phi(x, y) + J_{xy}W(\sigma_x, \xi_y))]\}| \\
& \leq \exp\left\{\beta \sum_{\substack{x \in \gamma_k \\ y \in \eta_j}} [\Phi^-(x, y) + \mathcal{J}W^-(\sigma_x, \xi_y)]\right\} - 1 \\
& \quad + \sum_{\substack{x \in \gamma_k \\ y \in \eta_j}} |1 - \exp\{-\beta(\Phi^+(x, y) + \mathcal{J}W^+(\sigma_x, \xi_y))\}| \\
& \leq \exp\left\{\beta[MLN(\gamma_k) + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k)]\right\} - 1 + LN(\gamma_k),
\end{aligned}$$

where the last inequality is obtained knowing that $\Phi^- \leq M$ and using the known bounds on the boundary conditions.

We also know, from assumption (A4) (see (21)), that

$$\int_{S^\gamma} \otimes_{x \in \gamma_k} g(d\sigma_x) \leq \int_{S^\gamma} \exp\{-a_V \sum_{x \in \gamma_k} |\sigma_k|^{q_V} + b_V N(\gamma_k)\} d\sigma_x. \quad (59)$$

Using assumption (21) and relations (57)-(59) we obtain

$$\mathfrak{D} \leq I_1 + I_2, \quad (60)$$

where

$$\begin{aligned}
I_1 &:= \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} \exp\left\{-\beta[H_k(\gamma_k) + E_{\gamma_k}(\sigma_k)] - a_V \sum_{x \in \gamma_k} |\sigma_k|^{q_V} + b_V N(\gamma_k)\right\} \\
&\times \left(\exp\left\{\beta[MLN(\gamma_k) + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k)]\right\} - 1\right) \otimes_{x \in \gamma_k} d\sigma_x \lambda(d\gamma_k)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &:= \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} LN(\gamma_k) \exp\{-\beta[H_k(\gamma_k) + E_{\gamma_k}(\sigma_k)] \\
&\quad - a_V \sum_{x \in \gamma_k} |\sigma_k|^{q_V} + b_V N(\gamma_k)\} \otimes_{x \in \gamma_k} d\sigma_x \lambda(d\gamma_k). \quad (61)
\end{aligned}$$

We start by estimating I_2 . We use again relations (57) and (58).

$$\begin{aligned}
I_2 \leq & \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} LN(\gamma_k) \exp \left\{ -\beta A_\Phi N(\gamma_k)^P + \beta B_\Phi N(\gamma_k) \right\} \\
& \times \exp \left\{ \beta \left[\mathcal{J}C_1 N(\gamma_k)^{2+\varepsilon^{-1}} + \mathcal{J}C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} \right] \right\} \\
& \times \exp \left\{ -a_V \sum_{x \in \gamma_k} |\sigma_k|^{q_V} + b_V N(\gamma_k) \right\} \otimes_{x \in \gamma_k} d\sigma_x \lambda(d\gamma_k)
\end{aligned}$$

In what follows, we will use Young's inequality in the following form

$$xy \leq \frac{x^a}{a\epsilon^a} + \frac{y^b \epsilon^b}{b}, \text{ for any } \epsilon > 0 \text{ and } a, b \text{ s.t. } \frac{1}{a} + \frac{1}{b} = 1. \quad (62)$$

Consider now only the factors depending on σ_x : $\exp\{\beta \mathcal{J}C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} - a_V \sum_{x \in \gamma_k} |\sigma_x|^{q_V}\}$. One can apply Young's inequality in the form (62) to obtain that

$$\beta \mathcal{J}C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} \leq c_1 + c_2 \sum_{x \in \gamma_k} |\sigma_x|^{q_V}$$

for some constants c_1 and c_2 . Choosing ϵ properly, one can make $c_2 < a_V$. Then there exist constants D_1 and \tilde{D}_1 , with $\tilde{D}_1 > 0$, such that

$$\begin{aligned}
& \exp\{\beta \mathcal{J}C_3 \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} - a_V \sum_{x \in \gamma_k} |\sigma_x|^{q_V}\} \\
& \leq \exp\{\beta \mathcal{J}N(\gamma_k) D_1\} \cdot \exp\{-\tilde{D}_1 \sum_{x \in \gamma_k} |\sigma_x|^{q_V}\}.
\end{aligned}$$

Take now the factors depending on $N(\gamma_k)$, including the one obtained in the right hand-side of the inequality above: $\exp\{-\beta A_\Phi N(\gamma_k)^P + N(\gamma_k)(\beta B_\Phi + b_V + \beta \mathcal{J}D_1) + \beta \mathcal{J}c_1 N(\gamma_k)^{2+\varepsilon^{-1}}\}$. In the same way as above, one can apply twice Young's inequality with properly chosen ϵ , to obtain

$$\exp\{-\beta A_\Phi N(\gamma_k)^P + N(\gamma_k)(\beta B_\Phi + b_V + \beta \mathcal{J}D_1) + \beta \mathcal{J}c_1 N(\gamma_k)^{2+\varepsilon^{-1}}\} \leq \exp D_2,$$

for some positive constant D_2 . Hence

$$I_2 \leq Le^{D_2} \int_{\Gamma_k \setminus \{\emptyset\}} N(\gamma_k) \int_{S^\gamma} \exp \left\{ -\tilde{D}_1 \sum_{x \in \gamma_k} |\sigma_x|^{q_V} \right\} \otimes_{x \in \gamma_k} dx \lambda(d\gamma_k).$$

Since $\exp\{-\tilde{D}_1 \sum_{x \in \gamma_k} |\sigma_x|^{q_V}\}$ is integrable with respect to the Lebesgue measure, there exists a constant D_3 such that

$$I_2 \leq D_4 \int_{\Gamma_k \setminus \{\emptyset\}} N(\gamma_k) \lambda(d\gamma_k) = D_4 \sum_{j=1}^{\infty} j \cdot \frac{z^j |Q_k|^j}{j!} = z D_4 |Q_k| e^{z|Q_k|}. \quad (63)$$

We proceed in a similar way to estimate I_1 , but first notice that

$$\begin{aligned} & \exp \left\{ \beta \left[MLN(\gamma_k) + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k) \right] \right\} - 1 \\ & \leq \beta \left[MLN(\gamma_k) + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k) \right] \cdot \\ & \quad \times \exp \left\{ \beta \left[MLN(\gamma_k) + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k) \right] \right\} \\ & \leq H_1 + H_2, \end{aligned}$$

where

$$\begin{aligned} H_1 := & \beta N(\gamma_k)(ML + \mathcal{J}L(L + C_W)) \exp\{\beta[MLN(\gamma_k) \\ & + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k)]\}, \end{aligned}$$

$$H_2 := \beta \mathcal{J} \exp\{\beta[MLN(\gamma_k) + \mathcal{J}L \sum_{x \in \gamma_k} |\sigma_x|^r + \mathcal{J}L(L + C_W)N(\gamma_k)]\}.$$

Therefore

$$\begin{aligned} I_1 \leq & \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} \exp\{-\beta[H_k(\gamma_k) + E_{\gamma_k}(\sigma_k)]\} H_1 \otimes_{x \in \gamma_k} d\sigma_x \lambda(d\gamma_k) \\ & + \int_{\Gamma_k \setminus \{\emptyset\}} \int_{S^\gamma} \exp\{-\beta[H_k(\gamma_k) + E_{\gamma_k}(\sigma_k)]\} H_2 \otimes_{x \in \gamma_k} d\sigma_x \lambda(d\gamma_k) \end{aligned}$$

Consider now the first integral, let us denote it by \mathcal{I}_1 . By separating the factors depending on σ_x and applying Young's inequality (62), and then doing the same for the factors depending on $N(\gamma_k)$, we can prove that there exists a constant D_5 such that

$$\mathcal{I}_1 \leq z D_5 |Q_k| e^{z|Q_k|}. \quad (64)$$

We can repeat this procedure for the second integral (denoted by \mathcal{I}_2) and obtain the estimate

$$\mathcal{I}_2 \leq D_6 J \int_{\Gamma_k \setminus \{\emptyset\}} \lambda(d\gamma_k) = \mathcal{J} D_6 |Q_k| (e^{z|Q_k|} - 1) \quad (65)$$

for some constant D_6 . Putting together (60), (63), (64) and (65) we obtain the estimate

$$\mathfrak{D} \leq z D_4 |Q_k| e^{z|Q_k|} + z D_5 |Q_k| e^{z|Q_k|} + \mathcal{J} D_6 |Q_k| (e^{z|Q_k|} - 1),$$

and the result follows, since we can take z and \mathcal{J} small enough to make \mathfrak{D} smaller than l . An application of the triangle inequality implies the required bound for more general boundary conditions. \square

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